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On the optimality criterion in structural control

A. Baratta*,† and O. Corbi‡

Department of 'Scienza delle Costruzioni', University of Naples 'Federico II', Naples, Italy

SUMMARY

The classical performance index optimization control algorithm is considered in order to check the real optimality of the control procedure; the basic steps for the optimal algorithm are reviewed, and the equation for the optimal control force derived. It is shown that the optimality conditions cannot be met with regard to the performance index, unless one is concerned with simple free oscillations. It is proved that in this case on one side the optimal control turns out to be of the linear closed-loop type, yielding explicit optimal control coefficients, and on the other side that no solution can exist of the optimal problem for a generic forcing function. It is concluded that one is forced to calibrate the control force for free oscillations, and that the reliability of the index procedure mainly rests on some implicit expectation that linear control can be calibrated in the absence of the external disturbance and that it works under forced oscillations as well. Furthermore, the problem of delayed active control, with reference to a s.d.o.f. system controlled by a closed-loop linear algorithm and under the action of a dynamic forcing function is investigated.

In particular, the effects produced on the response of the structure by the introduction in the control law of assessed critical values of time delay are analysed and the comparison is proposed between the numerical results that one gets by adopting two different procedures (on one hand the above-mentioned optimal linear control law and on the other hand the constrained minimization of the structural response norm) to compensate for time lag occurring in the actuation of the active control servomechanisms.

KEY WORDS: performance index; constrained minimization; norm optimal control; delayed control

INTRODUCTION

The problem to keep into admissible limits the entity of the response of a structure subject to seismic action can be approached by the application of active control techniques, based on the application of contrast forces able to limit the effects of dynamic shaking on the structure strength. When a control device is to be designed, a basic task is the choice of the algorithm, and the calibration of the relevant parameters, able to yield the optimal performance of the system.

* Correspondence to: A. Baratta, Departimento di Scienza delle Costruzioni, Università Degli Studi di Napoli Federico II, Piazzale V. Tecchio, 80 Fuorigrotta, Napoli 80125, Italy.
† Professor of Civil Engineering and Chair.
‡ Ph.D. student.
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The most popular technique to calibrate the control force is based on the optimal linear control, that essentially (see e.g. Soong [1]) consists of minimizing the so-called quadratic performance index under the constraint conditions defined by the equation of the motion and the initial conditions. Because of the end conditions, that cannot be met unless the forcing function is known *a priori*, a number of drastic simplifications are necessary; mainly the assumption of a linear control algorithm, and the annealing of the disturbance. Under the further assumption that the coefficients of the control force are constant the problem is reduced to solving the Riccati equation, that yields finally well-defined results. In the following, it is proved, from a theoretical point of view, that the performance index is not the best choice for a criterion to assess the effectiveness of a control algorithm under forced vibrations.

However, as this algorithm is often assumed because of its pretty good robustness that leads to results comparable with the ones one can get by adopting more complex and computationally heavier control laws, the formulation of the optimal linear control compensating for time delay is also considered. The exceeding of specified limits of the time gap elapsing between the sensors' reading of the excitation (open-loop control method) and/or (closed–open/closed-loop control method) of the response variables and the starting of the control actuation devices can, actually, vanify the control system efficiency with reference to structural response attenuation or, even, lead quickly to system instability.

In particular when, by means of extremely sophisticated hardware, one is allowed to think of small delay values, one could get an effective compensation by the introduction in the circuit of one or more elements (*phase lag compensators*) suitably calibrated in order to amplify the phase of the system transfer function. Theoretical and experimental studies [2] have tested the effectiveness of such methods in the steady-state field, as the introduction of time delay can actually, by reducing the system phase margin, lead to instability when this has rolled off to zero. Critical time delay values can, thus, immediately be determined for a given control law, by deducing them from the limit condition of instability occurrence, and phase-shift approaches to compensate for time lag effects produced on the stationary structural response can be adopted.

A different method to search for the optimal control parameters in the case of delay occurrence even in the transient phase, can be pursued by the optimization of the controlled response with the control force constrained under a prefixed threshold [3–5].

The reliability of this approach for critical delay values is studied, showing that the procedure, based on the search of the optimal values of the control coefficients by the mapping of the norm solution of the controlled system response, constitutes a really effective control method, transient-steady state reliable with respect to time-delay effects.

A comparison is then presented between the numerical results obtained by Hou and Iwan [6] by optimizing the performance index of the delayed control system for the identified critical delay values and the results one can obtain by searching for the minimum value of a suitably defined norm of the delay-controlled response.

OPTIMIZATION OF THE PERFORMANCE INDEX

The basic optimal linear control problem is to find the equation for the control force that is optimal with regard to the quadratic index of performance J, established, for a s.d.o.f., as follows:

Find the minimum of the functional

$$J(u(t), c(t)) = \int_0^T [k^4 u^2(t) + c^2(t)]\,dt \tag{1}$$

Subject to

$$\begin{aligned} &\ddot{u}(t) + 2\zeta_0\omega_0\dot{u}(t) + \omega_0^2 u(t) + c(t) = f(t) \\ &u(0) = u_0 \\ &\dot{u}(0) = \dot{u}_0 \end{aligned} \tag{2}$$

where superimposed dots denote time-derivatives, $u(t)$ is the structure's displacement variable, $f(t)$ and $c(t)$ are the external excitation and the control force, ζ_0 and ω_0 the damping coefficient and pulsation of the structure. Clearly the constraint condition (2) is the equation of the controlled structural motion. Building up the Lagrangian functional

$$\delta L(u, c, \lambda) = \int_0^T \{k^4 u^2(t) + c^2(t) + \lambda(t)[\ddot{u}(t) + 2\zeta_0\omega_0\dot{u}(t) + \omega_0^2 u(t) + c(t) - f(t)]\}\,dt \tag{3}$$

where λ is the Lagrange multiplier, the variations of the Lagrangian with respect to admissible variations $[\delta u(0) = \delta\dot{u}(0) = 0]$ of the basic variables lead to the necessary conditions for a minimum

$$\delta L_u(u, c, \lambda) = 0 \quad \forall \delta u \text{ admissible} \quad \rightarrow \quad \begin{bmatrix} \lambda(T) = \dot{\lambda}(T) = 0 \\ 2k^4 u(t) + \omega_0^2\lambda(t) - 2\zeta_0\omega_0\dot{\lambda}(t) + \ddot{\lambda}(t) = 0 \quad \forall t \in [0, T] \end{bmatrix} \tag{4}$$

$$\delta L_c(u, c, \lambda) = 0 \quad \forall \delta c \quad \rightarrow \quad 2c(t) + \lambda(t) = 0 \quad \forall t \in [0, T] \tag{5}$$

$$\delta L_\lambda(u, c, \lambda) = 0 \quad \forall \delta\lambda \quad \rightarrow \quad \ddot{u}(t) + 2\zeta_0\omega_0\dot{u}(t) + \omega_0^2 u(t) + c(t) - f(t) = 0 \quad \forall t \in [0, T] \tag{6}$$

Elimination of $\lambda(t)$ in Equations (4)–(6) yields the two o.d.e.'s with coupled end conditions

$$\begin{aligned} &\ddot{c}(t) - 2\zeta_0\omega_0\dot{c}(t) + \omega_0^2 c(t) - k^4 u(t) = 0 \\ &\ddot{u}(t) + 2\zeta_0\omega_0\dot{u}(t) + \omega_0^2 u(t) + c(t) = f(t) \\ &u(0) = u_0;\ \dot{u}(0) = \dot{u}_0 \\ &c(T) = 0 \\ &\dot{c}(T) = 0 \end{aligned} \tag{7}$$

which identify both the controlled response $u(t)$ and the relevant control force $c(t)$.

From Equations (7) one can obtain a single fourth-order o.d.e. in one of the unknowns, that, if $c(t)$ is chosen as the primary unknown and $T \rightarrow \infty$, can be written as

$$\begin{aligned} &\ddddot{c}(t) + 2\alpha^2\ddot{c}(t) + \beta^4 c(t) = k^4 f(t) \\ &\ddot{c}(0) - 2\zeta_0\omega_0\dot{c}(0) + \omega_0^2 c(0) = k^4 u_0 \\ &\dddot{c}(0) - 2\zeta_0\omega_0\ddot{c}(0) + \omega_0^2\dot{c}(0) = k^4\dot{u}_0 \\ &\lim_{t\rightarrow\infty} c(t) = 0 \\ &\lim_{t\rightarrow\infty} \dot{c}(t) = 0 \end{aligned} \quad \text{with} \quad \begin{aligned} \alpha^2 &= \omega_0^2(1 - 2\zeta_0^2) \\ \beta^4 &= \omega_0^4 + k^4 \end{aligned} \tag{8}$$

as in general $\zeta_0 \ll 1$ and then $\omega_0^2(1 - 2\zeta_0^2) > 0$.

FREE OSCILLATIONS: OPEN-LOOP CONTROL

Let us consider the case of free oscillation. For this purpose the general integral of the homogeneous equation associated to the first of Equations (8) has to be studied. For $f(t) \equiv 0$, the characteristic equation is

$$Z^4 + 2\alpha^2 Z^2 + \beta^4 = 0 \tag{9}$$

whose roots Z_i present real η and imaginary ω_r part in the form

$$\begin{aligned} \eta &= \psi\omega_0, \quad \psi = \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{k}{\omega_0}\right)^4} + 2\zeta_0^2 - 1} \\ \omega_r &= \rho\omega_0, \quad \rho = \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+\left(\frac{k}{\omega_0}\right)^4} + 1 - 2\zeta_0^2} \end{aligned} \tag{10}$$

With the above position and, after transforming in real, one gets

$$c(t) = e^{\eta t}[A\cos(\omega_r t) + B\sin(\omega_r t)] + e^{-\eta t}[C\cos(\omega_r t) + D\sin(\omega_r t)] \tag{11}$$

which yields the general integral of the homogeneous equation associated to the first of Equations (8) and therefore

$$\begin{aligned} c(0) &= A + C \\ \dot{c}(0) &= \eta A + \omega_r B - \eta C + \omega_r D = \eta(A - C) + \omega_r(B + D) \end{aligned} \tag{12}$$

The last two boundary conditions in Equation (8) can be met by cutting off the exponential terms with positive exponent (i.e. by putting $A = B = 0$), whence

$$c(t) = e^{-\eta t}[C\cos(\omega_r t) + D\sin(\omega_r t)] \tag{13}$$

Setting the constants C and D by introducing the first two boundary conditions of Equations (8) with $f(t) = 0$, one gets

$$\begin{aligned} &(\eta^2 + \omega_0^2 - \omega_r^2 + 2\zeta_0\omega_0\eta)C - (2\zeta_0\omega_0\omega_r + 2\omega_r\eta)D = k^4 u_0 \\ &(3\omega_r^2\eta - 2\zeta_0\omega_0\eta^2 + 2\zeta_0\omega_0\omega_r^2 - \eta^3 - \eta\omega_0^2)C + (\omega_r\omega_0^2 - \omega_r^3 + 3\omega_r\eta^2 + 4\zeta_0\omega_0\omega_r\eta)D = k^4\dot{u}_0 \end{aligned} \tag{14}$$

that completes the solution for the J-optimal control for free oscillations in the case of the open-loop control.

FREE OSCILLATIONS: CLOSED-LOOP LINEAR CONTROL

Coming back to the problem Equations (7) with $f(t) \equiv 0$ and $T \to \infty$, let search for possible solutions verifying (linear control)

$$c(t) = \bar{\omega}^2 u(t) + 2\mu\bar{\omega}\dot{u}(t) \tag{15}$$

with $\bar{\omega}$ and μ constants to be determined. Substitution of Equation (15) into the first of Equations (7) with $f(t)=0$ yields after some algebra

$$
\begin{aligned}
&2\mu\left[4\zeta_0\frac{\omega_0}{\bar{\omega}}+2\mu\right]\ddot{u}(t)+4(\mu\bar{\omega}+\zeta_0\omega_0)\dot{u}(t)+\left(\bar{\omega}^2+\frac{k^4}{\bar{\omega}^2}\right)u(t)=0\\
&\ddot{u}(t)+2[\zeta_0\omega_0+\mu\bar{\omega}]\dot{u}(t)+[\omega_0^2+\bar{\omega}^2]u(t)=0\\
&u(0)=u_0;\ \dot{u}(0)=\dot{u}_0\\
&\lim_{t\to\infty}c(t)=0\\
&\lim_{t\to\infty}\dot{c}(t)=0
\end{aligned}
\tag{16}
$$

The first and second equations of Equations (16) should be equivalent; hence the coefficients should be proportional according to any constant h to be determined

$$
2\mu\left[4\zeta_0\frac{\omega_0}{\bar{\omega}}+2\mu\right]=h,\quad 4(\mu\bar{\omega}+\zeta_0\omega_0)=2h[\zeta_0\omega_0+\mu\bar{\omega}],\quad \left(\bar{\omega}^2+\frac{k^4}{\bar{\omega}^2}\right)=h[\omega_0^2+\bar{\omega}^2] \tag{17}
$$

Solving, one obtains

$$
\begin{aligned}
&\bar{\omega}=\frac{8\zeta_0\mu\omega_0}{h-4\mu^2}\\
&\mu^2(16-8h)=2h^2-4h\\
&\frac{\bar{\omega}^4+k^4}{\omega_0^2\bar{\omega}^2+\bar{\omega}^4}=h
\end{aligned}
\tag{18}
$$

$$
\frac{2h^2-4h}{16-8h}\geqslant 0\to\begin{cases}\begin{bmatrix}2h^2-4h\geqslant 0\to h\in(-\infty,0)\cup(2,+\infty)\\ 16-8h\geqslant 0\to h\in(-\infty,2)\end{bmatrix}\Rightarrow h\leqslant 0\\ \begin{bmatrix}2h^2-4h\leqslant 0\to h\in(0,2)\\ 16-8h\leqslant 0\to h\in(2,+\infty)\end{bmatrix}\Rightarrow h=2\end{cases}
\tag{19}
$$

Since $h\leqslant 0$ is in conflict with the last equation in Equations (18), it only remains $h=2$, which yields the value of $\bar{\omega}$

$$
\bar{\omega}^2=-\omega_0^2\pm\sqrt{\omega_0^4+k^4} \tag{20}
$$

that, by eliminating the solution with negative sign, gives

$$
\bar{\omega}^2=\beta^2-\omega_0^2\geqslant 0\quad\text{with}\quad\beta^2=\sqrt{\omega_0+k^4} \tag{21}
$$

and as a consequence, the value of μ

$$
\mu^2=\frac{\bar{\omega}^2+4\zeta_0^2\omega_0^2\pm\sqrt{16\zeta_0^4\omega_0^4+8\zeta_0^2\omega_0^2\bar{\omega}^2}}{2\bar{\omega}^2} \tag{22}
$$

The algebra above proves that an optimal solution of the problem 1–2 with $f(t)=0$ for free oscillations exists in the form (15) (closed-loop linear control), and also yields in closed form the optimal coefficients of the algorithm.

FORCED OSCILLATIONS: OPEN-LOOP CONTROL

Search of the general integral

In the case of forced oscillations $[f(t) \neq 0]$ of Equations (8), the general integral for the optimal control force would be the sum of the general integral of the homogeneous equation, yielded by Equation (11), plus a particular integral. In order to evaluate the latter, consider the one with homogeneous initial conditions, i.e. $c(0) = \dot{c}(0) = \ddot{c}(0) = \dddot{c}(0) = 0$.

Considering the Laplace transform of the differential equation in Equations (8) one gets

$$(Z^4 + 2\alpha^2 Z^2 + \beta^4) C(Z) = k^4 F(Z) \tag{23}$$

with $C(Z)$ and $F(Z)$ the Laplace transforms of, respectively, $c(t)$ and $f(t)$, whence

$$C(Z) = \frac{1}{Z^4 + 2\alpha^2 Z^2 + \beta^4} k^4 F(Z) = H_4(Z) k^4 F(Z), \quad H_4(Z) = \frac{1}{Z^4 + 2\alpha^2 Z^2 + \beta^4} \tag{24}$$

Let Z_1, Z_2, Z_3, Z_4, be the roots of the characteristic polynomial $Z^4 + 2\alpha^2 Z^2 + \beta^4$, that is to say the poles of the system transfer function, so that

$$Z^4 + 2\alpha^2 Z^2 + \beta^4 = \prod_{i=1}^{4} (Z - Z_i) \tag{25}$$

Consider the L-transform of any linear combination of exponentials with exponents proportional to the roots

$$L\left[\sum_{i=1}^{4} a_i \mathrm{e}^{Z_i t}\right] = \sum_{i=1}^{4} \frac{a_i}{Z - Z_i} \tag{26}$$

By developing the sum at the rightward member, and considering that the roots of the characteristic polynomial are given by

$$\left.\begin{matrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{matrix}\right\} = \left\{\begin{matrix} \eta + \mathrm{i}\omega_\mathrm{r} \\ \eta - \mathrm{i}\omega_\mathrm{r} \\ -\eta - \mathrm{i}\omega_\mathrm{r} \\ -\eta + \mathrm{i}\omega_\mathrm{r} \end{matrix}\right. \tag{27}$$

with η and ω_r given by Equations (10), one gets by operating appropriate substitutions

$$\sum_{i=1}^{4} \frac{a_i}{\lambda - \lambda_1} = \Big[Z^3(a_1 + a_2 + a_3 + a_4) - Z^2(a_1 Z_3 + a_2 Z_4 + a_3 Z_1 + a_4 Z_2) + Z(a_1 Z_2 Z_4 + a_2 Z_1 Z_3 + a_3 Z_2 Z_4 + a_4 Z_1 Z_3) - (a_1 Z_2 Z_3 Z_4 + a_2 Z_1 Z_3 Z_4 + a_3 Z_1 Z_2 Z_4 + a_4 Z_1 Z_2 Z_3) \Big] \Big/ (Z^4 + 2\alpha^2 Z^2 + \beta^4) \tag{28}$$

Antitransform of the system transfer function $H_4(Z)$

For such transform to reproduce the function $H_4(Z)=1/(Z^4+2\alpha^2 Z^2+\beta^4)$ it is required

$$\begin{aligned}
&a_1+a_2+a_3+a_4=0\\
&a_1 Z_3+a_2 Z_4+a_3 Z_1+a_4 Z_2=0\\
&a_1 Z_2 Z_4+a_2 Z_1 Z_3+a_3 Z_2 Z_4+a_4 Z_1 Z_3=0\\
&a_1 Z_2 Z_3 Z_4+a_2 Z_1 Z_3 Z_4+a_3 Z_1 Z_2 Z_4+a_4 Z_1 Z_2 Z_3=-1
\end{aligned} \tag{29}$$

whose solution yields

$$\begin{aligned}
a_1&=-\frac{1}{8}\frac{\mathrm{i}}{\eta\omega_{\mathrm{r}}(\eta+\mathrm{i}\omega_{\mathrm{r}})}=-\frac{1}{8}\frac{\omega_r+\mathrm{i}\eta}{\eta\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}=-\frac{1}{8}\left[\frac{1}{\eta(\eta^2+\omega_{\mathrm{r}}^2)}+\mathrm{i}\frac{1}{\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}\right]=-E-\mathrm{i}F\\
a_2&=-\frac{1}{8}\frac{\mathrm{i}}{\eta\omega_{\mathrm{r}}(-\eta+\mathrm{i}\omega_{\mathrm{r}})}=\frac{1}{8}\frac{-\omega_r+\mathrm{i}\eta}{\eta\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}=-\frac{1}{8}\left[\frac{1}{\eta(\eta^2+\omega_{\mathrm{r}}^2)}-\mathrm{i}\frac{1}{\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}\right]=-E+\mathrm{i}F\\
a_3&=\frac{1}{8}\frac{\mathrm{i}}{\eta\omega_{\mathrm{r}}(\eta+\mathrm{i}\omega_{\mathrm{r}})}=\frac{1}{8}\frac{\omega_r+\mathrm{i}\eta}{\eta\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}=+\frac{1}{8}\left[\frac{1}{\eta(\eta^2+\omega_{\mathrm{r}}^2)}+\mathrm{i}\frac{1}{\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}\right]=E+\mathrm{i}F\\
a_4&=\frac{1}{8}\frac{\mathrm{i}}{\eta\omega_{\mathrm{r}}(-\eta+\mathrm{i}\omega_{\mathrm{r}})}=-\frac{1}{8}\frac{-\omega_r+\mathrm{i}\eta}{\eta\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}=+\frac{1}{8}\left[\frac{1}{\eta(\eta^2+\omega_{\mathrm{r}}^2)}-\mathrm{i}\frac{1}{\omega_{\mathrm{r}}(\eta^2+\omega_{\mathrm{r}}^2)}\right]=E-\mathrm{i}F
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
H_4(Z)=L\left[\sum_{i=1}^{4}a_i\mathrm{e}^{Z_i t}\right]&=L\{-2\mathrm{e}^{\eta t}[E\cos(\omega_r t)-F\sin(\omega_r t)]\\
&\quad+2\mathrm{e}^{-\eta t}[E\cos(\omega_r t)+F\sin(\omega_r t)]\}
\end{aligned} \tag{31}$$

Firstly, by the convolution theorem

$$c(t)=k^4\int_0^t h_4(t-\tau)f(\tau)\,\mathrm{d}\tau \tag{32}$$

with

$$h_4(x)=-2\mathrm{e}^{\eta x}[E\cos(\omega_r x)-F\sin(\omega_r x)]+2\mathrm{e}^{-\eta x}[E\cos(\omega_r x)+F\sin(\omega_r x)] \tag{33}$$

where, considered that both the constants E and F have definite values given by Equations (30), one recognizes an essentially unstable trend in the presence of the positive exponential term of the impulse response function.

The unstable component cannot be compensated by the free-oscillation component, unless the forcing function has some very particular character, even if the forcing function decays quickly to zero with time. This means that the two last end conditions in Equations (8) cannot be met, and no optimal solution exists for the performance index.

FORCED OSCILLATIONS: CLOSED-LOOP LINEAR CONTROL

By following a rationale analogous to that previously presented, it is possible to verify that the functional form (15) for the closed-loop control does not solve the optimal conditions for the performance index. In fact, in this case, substitution of Equation (15) into Equations (7) yields

$$\begin{aligned}
&2\mu\bar{\omega}\dddot{u}(t) + [\bar{\omega}^2 - 4\zeta_0\omega_0\mu\bar{\omega}]\ddot{u}(t) + 2[\omega_0^2\mu\bar{\omega} - \zeta_0\omega_0\bar{\omega}^2]\dot{u}(t) + [\omega_0^2\bar{\omega}^2 - k^4]u(t) = 0\\
&\ddot{u}(t) + 2(\zeta_0\omega_0 + \mu\bar{\omega})\dot{u}(t) + (\omega_0^2 + \bar{\omega}^2)u(t) = f(t)\\
&u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0\\
&c(T) = 0\\
&\dot{c}(T) = 0
\end{aligned} \tag{34}$$

By rearranging the second of Equations (34) and differentiating it one obtains the following expressions for the second and third time derivative of $u(t)$:

$$\begin{aligned}
\ddot{u}(t) &= f(t) - 2(\zeta_0\omega_0 + \mu\bar{\omega})\dot{u}(t) - (\omega_0^2 + \bar{\omega}^2)u(t)\\
\dddot{u}(t) &= \dot{f}(t) - 2(\zeta_0\omega_0 + \mu\bar{\omega})\ddot{u}(t) - (\omega_0^2 + \bar{\omega}^2)\dot{u}(t)
\end{aligned} \tag{35}$$

that, after substitution in the first of Equations (34), give

$$\begin{aligned}
&4\mu\bar{\omega}(2\zeta_0 + \mu\bar{\omega})\ddot{u}(t) + 4\bar{\omega}^2(\zeta_0\omega_0 + \mu\bar{\omega})\dot{u}(t) + (\bar{\omega}^4 + k^4)u(t) = 2\mu\bar{\omega}\dot{f}(t) + \bar{\omega}^2 f(t)\\
&\ddot{u}(t) + 2(\zeta_0\omega_0 + \mu\bar{\omega})\dot{u}(t) + (\omega_0^2 + \bar{\omega}^2)u(t) = f(t)\\
&u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0\\
&c(T) = 0\\
&\dot{c}(T) = 0
\end{aligned} \tag{36}$$

From the second of Equations (36) one can find that

$$4\bar{\omega}^2(\zeta_0\omega_0 + \mu\bar{\omega})\dot{u}(t) = 2\bar{\omega}^2 f(t) - 2\bar{\omega}^2\ddot{u}(t) - 2(\omega_0^2\bar{\omega}^2 + \bar{\omega}^4)u(t) \tag{37}$$

Equations (36) can, then, be rewritten as

$$\begin{aligned}
&(8\mu\zeta_0\bar{\omega}\omega_0 + 4\mu^2\bar{\omega}^2 - 2\bar{\omega}^2)\ddot{u}(t) + (k^4 - 2\omega_0^2\bar{\omega}^2 - \bar{\omega}^4)u(t) = 2\mu\bar{\omega}\dot{f}(t) - \bar{\omega}^2 f(t)\\
&\ddot{u}(t) + 2(\zeta_0\omega_0 + \mu\bar{\omega})\dot{u}(t) + (\omega_0^2 + \bar{\omega}^2)u(t) = f(t)\\
&u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0\\
&c(T) = 0\\
&\dot{c}(T) = 0
\end{aligned} \tag{38}$$

Considering that the second of Equations (38) coupled with the initial conditions [the third of Equations (38)] has a unique solution, it can be concluded that the optimal condition can be verified only if the forcing function belongs to the two-parameter (μ and $\bar{\omega}$) family of functions verifying the first equations in Equations (38), where the first member depends on the function $u(t)$ that is uniquely determined by the second equation. In other words, Equations (38) are a system of two independent differential equations that, for any choice of the two free parameters (μ and $\bar{\omega}$) yield only one unique solution and $f(t)$ should belong to the set of the relevant solutions.

DELAYED EQUATION OF THE MOTION

The equation of the motion for a s.d.o.f. controlled by means of a linear delayed control law can be written as follows:

$$m_0\ddot{u}(t) + c_0\dot{u}(t) + k_0 u(t) = \bar{f}(t) - c'\dot{u}(t-\tau) - k'u(t-\tau) \tag{39}$$

where superimposed dots denote time-derivatives, m_0, c_0 and k_0 are, respectively, the structural system mass, damping capacity and stiffness, $u(t)$ is the structure's displacement variable, c' and k' are the control system parameters, τ is the introduced time delay and $f(t) = \bar{f}(t)/m_0$ the external excitation. In standard form

$$\ddot{u}(t) + 2\zeta_0\omega_0\dot{u}(t) + \omega_0^2 u(t) = f(t) - 2\mu\bar{\omega}\dot{u}(t-\tau) - \bar{\omega}^2 u(t-\tau) \tag{40}$$

with ζ_0, ω_0 and μ, $\bar{\omega}$, respectively, the damping coefficient and frequency of the structure and the control force.

As the steady-state response of a delayed control system subjected to a harmonic excitation $f_0\,e^{i\omega_f t}$ of amplitude f_0 and frequency ω_f can be analysed by operating a Fourier transform of the equation of motion (Equation (39)) with the delayed control force one can deduce the amplitude of the steady-state response of the controlled system in the case of time-delay occurrence [6]

$$X_0 = \frac{f_0}{k\sqrt{(1-\delta^2+\lambda\cos(\omega_f\tau)+v\sin(\omega_f\tau))^2+(2\zeta_0\delta+v\cos(\omega_f\tau)-\lambda\sin(\omega_f\tau))^2}} \tag{41}$$

where

$$\delta = \frac{\omega_f}{\omega_0}, \quad \lambda = \frac{k'}{k_0}, \quad v = \frac{c'}{\omega_0 m_0} \tag{42}$$

Assume that the control parameters are chosen adopting any procedure for ideal synchronous control. The conditions for which the denominator of the amplitude X_0, for given structural characteristics and control parameters, becomes zero, define the resonance frequencies corresponding to the instability condition for the specific control law.

The critical values of time delay can then be identified as those for which the system phase reaches the instability phase condition

$$\omega_f\tau - \psi(\delta) = (2j+1)\pi \tag{43}$$

$$\psi(\delta) = \arctan\frac{(1-\delta^2)v - (2\zeta\delta)\lambda}{(1-\delta^2)\lambda + (2\zeta\delta)v} \tag{44}$$

where j is an arbitrary integer and ψ the system phase.

The two conditions correspond, in terms of Bode representation system transfer function, to the case when the phase of the synchronously controlled system reaches -180° at the frequency where the logarithmic gain of the system transfer function is equal to zero (the cross-over frequency ω_{cf}), that is to say when the Phase-Margin (PM) becomes zero. The phase reduction introduced by time delay in control actuation produces the following decrease of the system PM [7]

$$\mathrm{PM} = 180^\circ + \alpha_{cf} \rightarrow \mathrm{PM} = 180^\circ + \alpha_{cf} - \omega_{cf}\tau \tag{45}$$

α_{cf} being the system phase at the crossover frequency ω_{cf}.

IDENTIFICATION OF THE OPTIMAL CONTROL PARAMETERS

Optimal linear control

An optimal linear control law is adopted by Iwan and Hou [6] to find the synchronous control coefficients for an undamped s.d.o.f. with unit mass and stiffness.

For an n-degree-of-freedom system, the procedure requires the solution of the following problem:

$$J_0 = \min J([u], [\bar{c}])$$

$$\text{sub}\begin{cases}\{A\}[u(t)] + \{B\}[\bar{c}(t)] + \{D\}[\bar{f}(t)] - [\dot{u}(t)] = 0\\ [u(0)] = [u_0]\end{cases} \tag{46}$$

where J is the quadratic index of performance, $[u(t)]$ the state vector, $[\bar{f}(t)]$ and $[\bar{c}(t)]$ are the excitation and control vector and $\{A\}$, $\{B\}$, $\{D\}$ suitably defined state matrices. In the following J is assumed to have the form

$$J = \int_0^T \left\{\frac{R_s}{2}(k_0 u^2 + m_0 \dot{u}^2) + R_f(k' u + c' \dot{u})^2\right\} dt \tag{47}$$

R_s and R_f being unit weight coefficients.

The individuated critical values of time delay for the resulting control law expression have then been used to search for new parameters that minimize, for a given forcing function, the performance index rewritten, in the case of the steady-state response, as follows:

$$J = R_s X_0^2 + R_f(\lambda^2 + v^2) \tag{48}$$

Norm optimization control

A different approach for the localization of the optimal control coefficients can be adopted by expressing the controlled response function by means of the norm of the response parameters rather than by the determination of the detailed system response; this way the control algorithm results to be independent on the forcing function characteristics.

By expressing the norm of the displacement function $u(t)$ by the norm of the impulsive response function $h(t)$ and of the excitation $f(t) = \bar{f}(t)/m_0$, one can find an upper bound for the maximum displacement that occurs and hence for the maximum shear force $S(t)$ and maximum control force $c(t) = \bar{c}(t)/m_0$

$$u_{\max} \leqslant \| f(t)\| L(\omega, \zeta), \quad S_{\max} \leqslant \omega^2 \| f(t)\| L(\omega, \zeta), \quad c_{\max} \leqslant \omega_c^2 \| f(t)\| H(\omega', \zeta') \tag{49}$$

with $\| f(t)\|$ the excitation norm and $H(\omega', \zeta')$ and $L(\omega, \zeta)$ the norm operator of the control force and of the response, this one being valid for the uncontrolled case, with $\zeta = \zeta_0$ and $\omega = \omega_0$, as well as the controlled one with $\zeta = \zeta_c$ and $\omega = \omega_c$

$$\omega_c = \sqrt{\omega_0^2 + \omega^2}, \quad \zeta_c = \frac{\zeta_0 \omega_0 + \mu\bar{\omega}}{\omega_c}. \tag{50}$$

As the norms valid for the controlled case, L_c and H, have the form

$$L_c^2(\bar{\omega},\mu)=\int_0^{\infty} h_c^2(x)\,\mathrm{d}x,\quad H^2(\bar{\omega},\mu)=\int_0^{\infty}\left(2\frac{\mu}{\bar{\omega}}\dot{h}_c(x)+h_c(x)\right)^2\mathrm{d}x \tag{51}$$

the problem can then be set as follows:

$$\text{FIND}\quad \min_{\bar{\omega},\mu} L\quad \text{SUB}\quad C\leqslant C_0 \tag{52}$$

with
$$L=S_{\max}/\|f\|,\quad C=c_{\max}/\|f\|,\quad C_0=\gamma S_0, \tag{53}$$

γ being a prefixed percentage of the uncontrolled maximum shear force S_0.

For the search of the optimal control parameters for the specified critical values of time delay, an analogous procedure is followed. By suitably adimensionalizing the equation of the motion in form (40)

$$\begin{aligned}&\vartheta=\omega_0 t,\quad \overline{\vartheta}=\omega_0\tau,\quad (*)'=d(*)/d\vartheta\quad y(\vartheta)=u(\vartheta/\omega_0);\\&y''(\vartheta)+2\zeta y'(\vartheta)+y(\vartheta)+2\eta y'(\vartheta-\overline{\vartheta})+\alpha y(\vartheta-\overline{\vartheta})=\varphi(\vartheta)\\&y(0)=y_0;\ y'(0)=y_0'\end{aligned} \tag{54}$$

where α and η are the redefined adimensional control parameters, and $\gamma(\vartheta)$ and $\varphi(\vartheta)$ the adimensionalized system response and forcing function, a discretization is operated and a step-by-step procedure [5] is adopted to invidaute the upper bounds of the system displacements, shear force and control force

$$s_{\max}=y_{\max}\leqslant\|L_c(\eta,\alpha)\|\cdot\|\varphi\|\quad c_{\max}\leqslant C(\eta,\alpha)\|\cdot\|\varphi\| \tag{55}$$

with the energy of the forcing function $\varphi(\vartheta)$ and the norms of the response operator and of the control force

$$\begin{aligned}&\|\varphi(\vartheta)\|_{\vartheta_i}^2=\int_0^{\vartheta_i}\varphi^2(\vartheta)\,\mathrm{d}\vartheta\approx\sum_{j=1}^{i}\varphi_j^2\Delta\vartheta\,\|L_c(\eta,\alpha)\|_{\vartheta_i}^2\approx\left[\sum_{j=1}^{i}h_{ij}^2\Delta\vartheta\right],\\&\|C(\eta,\alpha)\|^2=\left[\Delta\vartheta\sum_{j=1}^{i-k}(2\eta h_{ij}'+\alpha h_{ij})^2\right]\end{aligned} \tag{56}$$

$\Delta\vartheta$ being the time distance between the elements ϑ_i of a discrete sequence of i instants on the duration of the external disturbance and h_{ij} the elements of the discretized form of the impulsive response function iteratively determined.

Then the whole problem can be defined as follows

$$\text{FIND}\quad \min_{\eta,\alpha} L_c\quad \text{SUB}\quad C\leqslant C_0 \tag{57}$$

where

$$\|S(\eta,\alpha)\|=\|L_c(\eta,\alpha)\|=L_c(\eta,\alpha/\zeta_0),\quad \|C(\eta,\alpha)\|=C(\eta,\alpha/\zeta),\quad C_0=\gamma\, S(0,0/\zeta_0) \tag{58}$$

γ being an arbitrary integer expressing the percentage of the norm $S(0,0/\zeta_0)$ of the uncontrolled system shear force.

NUMERICAL RESULTS

For an undamped structural system with unit mass and stiffness, a comparison has been provided between the numerical results obtained by Hou and Iwan by following the procedure previously illustrated and the results one can obtain by adopting the norm optimization control method.

The case of ideal synchronous control has been considered and, for the obtained values of optimal parameters, the families of critical delay values investigated to check the non-effectiveness of the control algorithm whereas such a delay is introduced in the system. Of course, the structural response is highly unstable for such critical values, as one can observe by looking at Figure 1.

The phase-shift approach, proposed for compensation of the steady-state effects produced by time delay, is based on the introduction in the system of elements able to provide an increase of the system phase; the derivative part of the linear control law represents actually a phase compensator whose importance appears, hence, considerable. By the phase-shift approach, the original control gains are modified such that both the real system and the ideal system have the same active mechanical characteristics.

From Figure 1 the norm optimization control algorithm appears clearly to be reliable even in the correspondence of the introduced critical delay value $\tau = 0.75\,\mathrm{s}$; the steady-state response obtained with the time-delay compensation performed by means of the quadratic index in the form (48)—practically strictly closed to the one obtained for the synchronous control—would seem to be better than the response deduced by searching for the optimal control parameters for the specified time delay. Actually, while the norm minimization provides control coefficients suitable for every forcing function and thus permit with these only two values to draw the whole

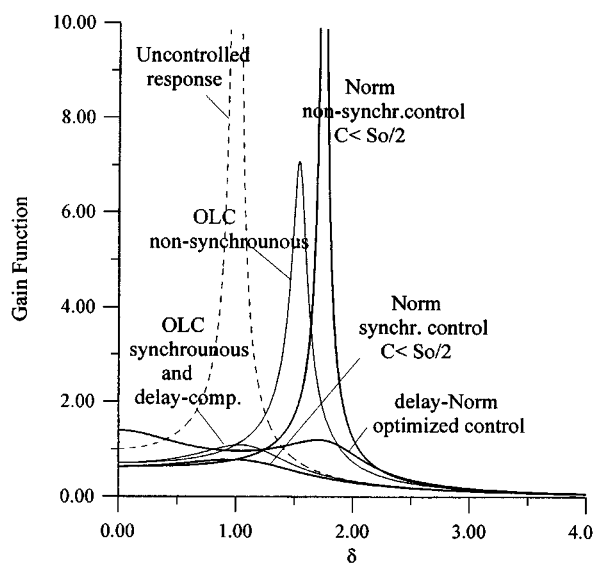

Figure 1. Amplitude function of the steady-state response for $\tau = 0.75\,\mathrm{s}$ versus the ratio δ.

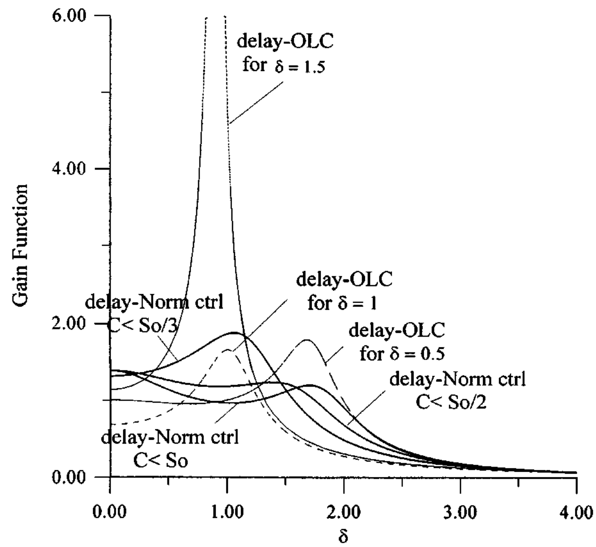

Figure 2. Uncontrolled and controlled time response to a sine wave with $\omega_f = 1\,s^{-1}$ for $\tau = 0.75$ s. U_0 = Uncontrolled response; U_{OLC} = Optimal Linear Controlled response; U_{NOC} = Norm Optimization Controlled response.

corresponding amplitude curve, the minimization of the index J in the form depending on the system gain function, is linked to the knowledge of the excitation frequency and, thus, presents varying control parameters for each point of its amplitude curve.

By applying the control parameters deduced with the performance criterium under the specified delay for particular values of ω_f to the whole range of possible excitation frequencies, the effectiveness of the compensated OLC algorithm for δ values not corresponding to the one for which the control system has been adjusted, has been analysed.

A direct comparison can be made between the OLC gain for $\omega_f = 1\,s^{-1}$ and the Norm control gain with $C < S_0$ as the energy of the supplied control force is identical for the two cases.

The time response of the delay-controlled structural system has been observed; for a harmonic excitation and low values of delay occurring in the control system, the two control procedures are almost equivalent as shown in Figure 2 for $\tau = 0.75$ s. With an higher time delay the OLC algorithm seems to fail, as illustrated for $\tau = 3$ s in terms of time response in Figure 3 and of gain function in Figure 4.

In Figure 5 the curves obtained by the control parameters valid for $\omega_f = 1\,s^{-1}$ and $\omega_f = 0.5\,s^{-1}$ seems to work quite well, whilst the ones valid for $\omega_f = 1.5\,s^{-1}$ lead quickly the system to instability. Even for $\omega_f = 0.5\,s^{-1}$ and $\omega_f = 1\,s^{-1}$ the peaks of the OLC gain function are higher than those relative to the case of norm optimization.

The simulations of structural response to a white noise show, in Figure 6(a) and 6(b), respectively, for $\tau = 0.75$ s and $\tau = 3$ s how important the OLC response is when the forcing function does not have harmonic form; attention has to be paid to the fact that one has considered an energy of the control force identical for the two control procedures in the case of $\tau = 0.75$ s, whilst,

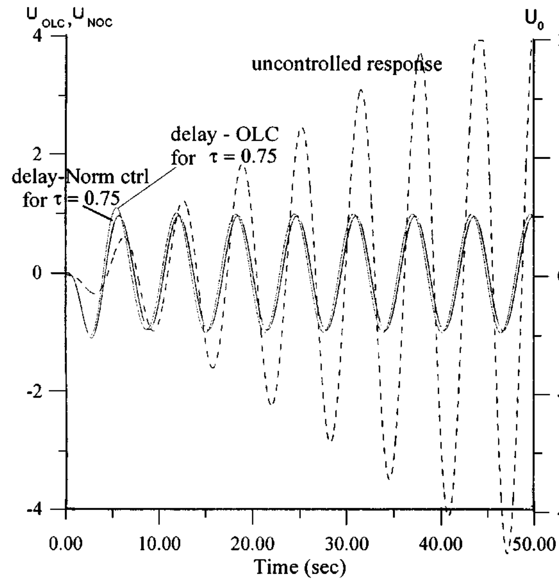

Figure 3. Uncontrolled and controlled time response to a sine wave for $\tau = 3$ s. U_0 = Uncontrolled response; U_{OLC} = Optimal Linear Controlled response; U_{NOC} = Norm Optimization Controlled response.

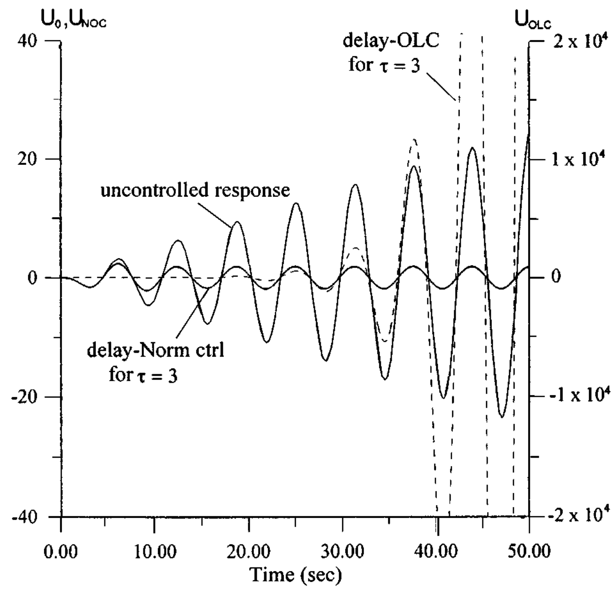

Figure 4. Amplitude function of the steady-state response for $\tau = 3$ s versus the ratio δ. U_0 = Uncontrolled response; U_{OLC} = Optimal Linear Controlled response; U_{NOC} = Norm Optimization Controlled response.

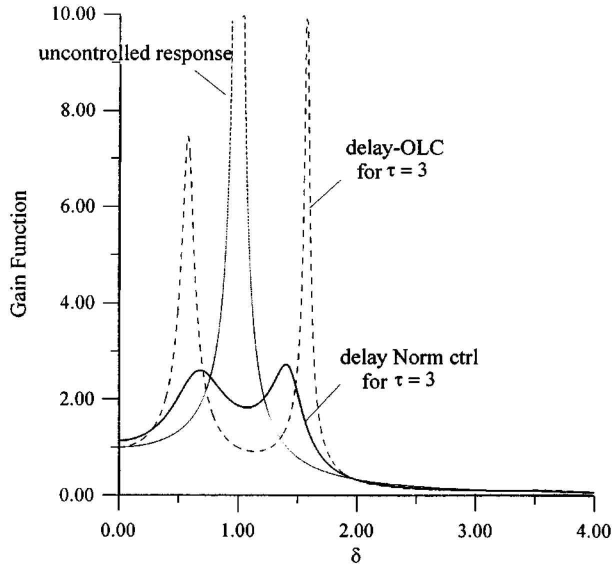

Figure 5. Amplitude function of the steady-state response for $\tau = 0.75$ s versus the ratio δ.

Table I. Statistical analysis of the controlled response for $\tau = 0.75$ s and $\tau = 3$ s.

Control	μ_x	μ_y	μ_c	τ	μ_x	μ_y	μ_c	τ
OLC	0.6712	0.2488	0.3960	0.75	0.6712	0.6166	0.8023	3.00
Norm Optimiz. Ctrl.	0.6712	0.2320	0.3587	0.75	0.6712	0.33967	0.21613	3.00

for $\tau = 3$ s, it is, for the Norm Optimization Control method, equal to one third of that relative to the OLC method.

To test the response of the structural system controlled with each of the two algorithms under more than only one particular excitation a statistical analysis has been carried out. A sample consisting of a hundred of random forcing functions has been considered to deduce some statistically significant parameters of the structural response.

Table I illustrates, for the two different values of time delay previously introduced, the averages μ_x of the uncontrolled response, μ_y of the controlled system by adopting OLC and Norm Optimization Control, and μ_c of the control force. The obtained numerical results show that, for small time delays, a more effective control with a lower energy expense is achieved when adopting the method of constrained minimization of the response norm operator and that this effect is enhanced when bigger values of time lag are introduced in the control system.

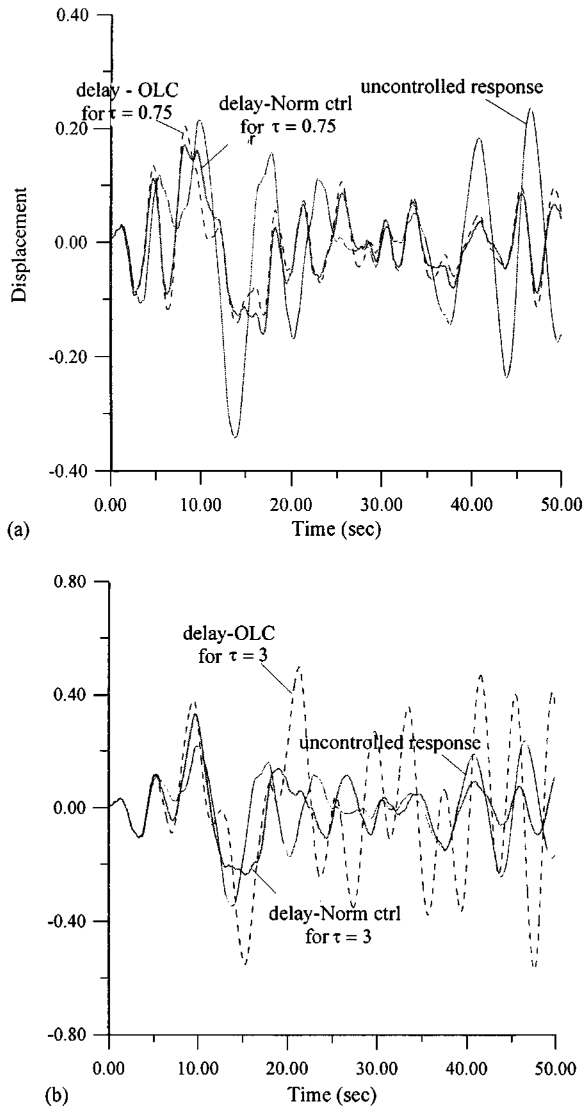

Figure 6. Uncontrolled and controlled time response to a white noise: (a) for $\tau = 0.75$ s, (b) for $\tau = 3$ s.

CONCLUSIONS

The developments in the paper aim at clarifying some aspects of the classical performance index theory in calibration of linear control algorithms is commonly made use of. It is pointed that no optimal solution exists for the performance index under an external disturbance, even if this is *a priori* well known, in that all solutions of Equations (8) contain some diverging component that is in conflict with the end conditions on $c(t)$, unless $f(t)$ has some very particular functional form.

Therefore, if one wants to use the optimal formulation for the performance index, one is forced to make recourse to the idea that the same control strategy works for forced oscillations as well as for free oscillations, and to assume the solution for the latter. The optimality character of the performance index for forced oscillations cannot be guaranteed. One might conclude for the opportunity to address other optimality criteria (e.g. [3]) whose performance, when closed loop algorithms are assumed, is intrinsically independent on the forcing function form.

A comparison has been presented between two different approach of search of the optimal control parameters under the hypothesis of time delay occurrence in the control system. The reliability of the linear control algorithm that provides a constrained minimization of the response operator norm is tested with reference to some critical values of time delay.

Simulations and statistical analysis of the controlled structural response point out that the Norm Optimization procedure represents a really effective control method, transient-steady-state reliable with respect to effects of even higher time delay values. Furthermore the particular control algorithm is able to provide a better structural response attenuation in comparison to the one deriving from the adoption of a delay compensated OLC approach, whilst accomplishing the purpose of strongly containing the control force energy.

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